TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 342, Number 1, March 1994

THE H^2 CORONA PROBLEM AND $\overline{\partial}_b$ IN WEAKLY PSEUDOCONVEX DOMAINS

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ABSTRACT. We derive a Bochner-Kodaira-Nakano-Morrey-Kohn-Hörmander type equality in holomorphic vector bundles and obtain L^2-estimates for $\overline{\partial}_b$ in a pseudoconvex domain that admits a plurisubharmonic C^2 defining function. We combine these with the trick in Wolff's proof of the corona theorem and obtain a H^2-corona theorem in such a domain.

0. INTRODUCTION

Let D be a domain in $\mathbb{C}^n$ and $g_i \in H^\infty(D)$ such that

$$\sum_1^k |g_i|^2 \geq \delta^2 > 0. \tag{1}$$

The problem whether there are $u_j \in H^\infty(D)$ such that $\sum_1^k g_i u_i = 1$ is known as the corona problem. The answer is affirmative in e.g. all finitely connected domains in $\mathbb{C}$, but unknown even in the ball if $n > 1$. In [11 and 5] are constructed smooth domains in $\mathbb{C}^3$ and $\mathbb{C}^2$ which have strictly pseudoconvex boundary in all but one point, but in which the corona theorem fails. However it follows from [13] that in case of two generators, i.e. $k = 2$, in a strictly pseudoconvex domain there is a solution in BMO.

It is clear that if the corona problem is solvable, then to any $q \in H^2(D)$ there are $u_j \in H^2(D)$ such that $\sum g_j u_j = q$. In this paper we prove such a theorem in a pseudoconvex domain D admitting a C^2 plurisubharmonic defining function ρ, i.e. ρ be of class C^2 in a neighborhood of $\overline{D}$, $D = \{\rho < 0\}$, $d\rho \neq 0$ on ∂D and $i\partial\overline{\partial}\rho \geq 0$ in D. In particular, the domains in [11 and 5] are of this kind. However, there are examples of pseudoconvex C^2 domains without plurisubharmonic defining function, see [3].

Theorem 1. *Suppose D is a pseudoconvex domain in $\mathbb{C}^n$ with a C^2 plurisubharmonic defining function. Let g be a $j \times k$-matrix of functions in $H^\infty(D)$ such that*

$$\det gg^* \geq \delta^2 > 0. \tag{2}$$

Received by the editors November 27, 1990 and, in revised form, January 17, 1992.
1991 *Mathematics Subject Classification.* Primary 32A10, 32A35, 32F15, 32F20.
Key words and phrases. Corona theorem, $\overline{\partial}_b$-equation.

0002-9947/94 $1.00 + $.25 per page

Then to every j*-column* q *of functions in* $H^2(D)$ *there is a* k*-column* u *in* $H^2(D)$ *such that*

$$gu = q \tag{3}$$

and $\|u\|_{H^2} \le C_\delta \|q\|_{H^2}$. *If* $j = 1$ *one can take* $C_\delta = C_\varepsilon/\delta^{1+\varepsilon+\min(n,k-1)}$ *where* C_ε *only depends on* $\|g\|_\infty$, n, k *and* $\varepsilon > 0$.

Note that (2) implies that g is surjective and hence that (3) is pointwise solvable. In fact it is enough to assume that g has constant rank, the product of the nonzero eigenvalues of gg^* are bounded by δ^2 from below and that (3) is pointwise solvable to get the conclusion of Theorem 1, cf. [1].

Since any smooth pluriharmonic χ on $\overline{D}$ has the form $\chi = \log|f|^2$ for some nonvanishing holomorphic f (at least if $H^1(\overline{D}, \mathbb{C}) = 0$), we can apply Theorem 1 to q/f instead of q and obtain a solution to (3) such that

$$\int_{\partial D} |u|^2 e^{-\chi} \le c_\delta^2 \int_{\partial D} |q|^2 e^{-\chi}. \tag{4}$$

If $n = 1$, χ can be freely chosen in (4) and this implies, see [2] or [1], that there is a bounded solution u such that $\|u\|_\infty \le c_\delta \|q\|_\infty$ if q is bounded. Hence Theorem 1 is equivalent to the corona theorem in the unit disc.

Theorem 1′. *Let* D *be as in Theorem* 1. *If* q *and* $g = (g_1, \dots, g_k)$ *in* $H^\infty(D)$ *satisfy*

$$|q| \le |g|^{1+\varepsilon+\min(n,k-1)}, \tag{5}$$

then there is a solution u *to* (3) *in* $H^2(D)$ *such that*

$$\int_{\partial D} |u|^2 e^{-\chi}\, dS \le C_\varepsilon \int_{\partial D} e^{-\chi}\, dS.$$

Again, for $n = 1$, this implies that there is a bounded solution if $|q| \le |g|^{2+\varepsilon}$. This also follows from Wolff's proof of the corona theorem, see [7].

Corollary. *If* $q = q_1 q_2$ *where* q_1 *satisfies* (5) *and* $q_2 \in H^2(D)$ *is nonvanishing, then there is a solution* u *to* (3) *such that*

$$\int_{\partial D} |u|^2\, dS \le C_\varepsilon \int_{\partial D} |q_2|^2\, dS.$$

Our proof of Theorem 1 (and 1′), as most proofs of the corona theorem, is based on an estimate of solutions of a $\overline{\partial}_b$-equation. This approach was first introduced by Hörmander in [10]. Using the Koszul complex one can, in principle, reduce the theorem to systems of (scalar-valued) $\overline{\partial}$- (or $\overline{\partial}_b$-) equations, at least if $j = 1$. However, unless $n = 1$ or $k \le 2$, one has to solve, successively, a sequence of $\overline{\partial}$-equations (involving forms of higher bidegree) and this seems to lead to considerable difficulties. Instead we reformulate the theorem as a $\overline{\partial}_b$-problem in a holomorphic vector bundle, following the lines in [12] (but with $\overline{\partial}$ replaced by $\overline{\partial}_b$), in which L^2-estimates for division problems are treated in a very general setting.

The $\overline{\partial}_b$-equation is treated by a generalization of a variant due to Berndtsson [2], of the Morrey-Kohn-Hörmander identity (see §3). In [2], this new identity was used to get $L^2(\partial D)$-estimates for the (scalar-valued) $\overline{\partial}_b$-equation. Since we need estimates for $\overline{\partial}_b$ in a vector bundle, our first aim is to generalize to this case. This leads to a $L^2(\partial D)$-theorem for $\overline{\partial}_b$ in a holomorphic vector bundle over a pseudoconvex domain in a Kähler manifold, see Theorem 2 in §4.

However, in order to prove Theorem 1 (and Theorem 1$'$) we cannot use Theorem 2 directly since it just deals with a size estimate of the right-hand side. One also has to take into account an appropriate estimate of derivatives. This is the trick introduced by Wolff in his proof of the corona theorem, see [7].

The paper is divided into six sections. After some necessary preliminaries in §1, we discuss the $\overline{\partial}_b$-equation in §2. In §3 we derive the above-mentioned equality and show its connection to the Morrey-Kohn-Hörmander equality as well as the Bochner-Kodaira-Nakano equality. In §4 we prove our L^2-estimate for $\overline{\partial}_b$ (Theorem 2) and in the remaining two sections we prove Theorem 1 and Theorem 1$'$.

1. Notational preliminaries

Let X be a Kähler manifold with fundamental form ω, so that $dV = \omega^n/n!$ is the volume measure on X, and let $D = \{\rho < 0\}$ be a relatively compact domain in X, where ρ is smooth and $d\rho \neq 0$ on ∂D. We give ∂D the orientation so that a $(2n-1)$-form α is oriented on ∂D if and only if $d\rho \wedge \alpha$ is oriented on X. Then the surface measure dS on ∂D is given by $dS = d\rho/|d\rho| \,\lrcorner\, dV$ on ∂D, where inner multiplication $\lrcorner$ for forms α and β is defined by

$$\langle \alpha \lrcorner \beta, \gamma \rangle = \langle \beta, \overline{\alpha} \wedge \gamma \rangle. \tag{1}$$

Here $\langle\ ,\ \rangle$ is the induced inner product for forms, i.e. if $*$ is the complex-linear Hodge star operator, then $\langle \alpha, \beta \rangle\, dV = *(\alpha \wedge *\overline{\beta})$. Note that $** = (-1)^{p+q}$ on (p, q)-forms (since X has even real dimension) and that $\alpha \lrcorner \beta = *(\alpha \wedge *\beta)$ for any 1-form α. If α is a $(2n-1)$-form, we have that

$$\int_{\partial D} \alpha = \int_{\partial D} *(d\rho \wedge \alpha)\, dS/|d\rho|. \tag{2}$$

To see (2), notice that $d\rho \wedge *(d\rho \wedge \alpha)\, dS/|d\rho| = *(d\rho \wedge \alpha)\, d\rho \wedge dS/|d\rho| = *(d\rho \wedge \alpha)\, dV = d\rho \wedge \alpha$, and hence both integrands in (2) are equal, considered as forms on ∂D.

If $E \to X$ is a holomorphic vector bundle over X with hermitian metric $\langle\ ,\ \rangle_E$, then we get a metric for $\xi, \eta \in \mathscr{E}_{p,q}(X, E)$ $(= \mathscr{E}(X, \Lambda^{p,q} T^* \otimes E)$ i.e. smooth E-valued (p, q)-forms) by putting $\langle \alpha \cdot s, \alpha' \cdot s' \rangle = \langle \alpha, \alpha' \rangle \langle s, s' \rangle_E$ on composite elements. If $s \to s'$ is the conjugate linear mapping from E to its dual bundle E^*, such that $s' \cdot s = \langle s, s \rangle_E$, and $\overline{*}\colon \mathscr{E}_{p,q}(X, E) \to \mathscr{E}_{q,p}(X, E^*)$ by putting $\overline{*}(\alpha \cdot s) = *\overline{\alpha} \cdot s'$ on composite elements, then

$$\langle \xi, \eta \rangle\, dV = \xi \wedge \overline{*}\eta, \qquad \xi, \eta \in \mathscr{E}_{p,q}(X, E). \tag{3}$$

Moreover, $\alpha \lrcorner \xi = \overline{*}(\overline{\alpha} \wedge \overline{*}\xi)$, cf. (5), if α is a 1-form and ξ any E-valued form.

For $\xi, \eta \in \mathscr{E}_{p,q}(X, E)$, one of which has compact support, we have the inner product

$$(\xi, \eta) = f\langle \xi, \eta \rangle\, dV = \int \xi \wedge \overline{*}\eta.$$

Let $D = D' + D''$ denote the Chern connection on E as well as on E^* (with respect to $\langle\ ,\ \rangle_E$). Then the formal adjoints $(D')^*$ and $(D'')^*$ are given by $(D'')^* = -\bar{*}D''\bar{*}$, $(D')^* = -\bar{*}D'\bar{*}$.

If s is a holomorphic section to E then $D''(\alpha \cdot s) = (\bar\partial\alpha)\cdot s$ and therefore we sometimes write $\bar\partial$ instead of D''.

2. The $\bar\partial_b$-equation

Using the notation from §1 we have

Proposition 1. *Suppose $\eta \in \mathscr{E}_{p,q+1}(\overline{D}, E)$ and $\xi \in \mathscr{E}_{p,q}(\overline{D}, E)$. Then*

$$(1)\qquad \int_D \langle D''\xi, \eta\rangle\, dV - \int_D \langle \xi, (D'')^*\eta\rangle\, dV = \int_{\partial D} \langle \xi, \partial\rho \lrcorner \eta\rangle\, dS/|d\rho|.$$

Remark. Throughout this paper $(D'')^*$ denotes the *formal* adjoint of D''. When dealing with the D''-Neumann problem $(D'')^*$ is an operator with a specified domain $\operatorname{dom}(D'')^*$. For instance, (1) implies that $\eta \in \mathscr{E}_{p,q}(\overline{D}, E)$ is in $\operatorname{dom}(D'')^*$ if and only if $\partial\rho \lrcorner \eta|_{\partial D} = 0$.

Sketch of proof. By (1), (3), and (2) in §1 we have that

$$\int_{\partial D} \langle \xi, \partial\rho \lrcorner \eta\rangle\, dS/|d\rho| = \int_{\partial D} *(\bar\partial\rho \wedge \xi \wedge \bar{*}\eta)\, dS/|d\rho| = \int_{\partial D} \xi \wedge \bar{*}\eta,$$

so by Stokes' theorem and bidegree reasons the right-hand side of (1) equals $\int_D d(\xi \wedge \bar{*}\eta) = (D''\xi, \eta) - (\xi, (D'')^*\eta)$. □

Let $f \in \mathscr{E}_{n,q+1}(\overline{D}, E)$. We say that an E-valued form (current) u on ∂D solves $\bar\partial_b u = f$ if

$$(2)\qquad \int_{\partial D} \langle u, \partial\rho \lrcorner \alpha\rangle dS/|d\rho| = \int_D \langle f, \alpha\rangle\, dV$$

for all $\alpha \in \mathscr{E}_{n,q+1}(\overline{D}, E)$ such that $(D'')^*\alpha = 0$.

Hence, by (1), u must have bidegree (n, q) in the sense that $\alpha \wedge \bar\partial\rho$ has bidegree $(n, q+1)$, and $\bar\partial f = 0$ (since one can take $\alpha = \bar{*}\psi$ for any $\bar\partial$-closed ψ). Notice that

$$(3)\qquad \|u\|^2_{\partial D} = \int_{\partial D} |u \wedge \bar\partial\rho|^2\, dS/|\partial\rho|_3$$

defines a norm for the space of smooth (n, q)-forms on ∂D, and let $L^2_{n,q}(\partial D, E)$ be its completion with respect to this norm.

Remark. To be more precise, any $u \in \mathscr{E}(\partial D, \Lambda^{n,q}T(X)\otimes E)$ has an orthogonal decomposition $u = u_1 + u_2$, where $\bar\partial\rho \wedge u_1 = 0$ and $\bar\partial\rho \lrcorner u_2 = 0$. Thus (n, q)-forms (currents) V such that $\partial\rho \lrcorner V = 0$ can be isometrically identified with intrinsic (n, q)-forms (currents) on ∂D.

Proposition 2. *Let $f \in \mathscr{E}_{n,q+1}(\overline{D}, E)$ be D''-closed. Then $\bar\partial_b u = f$ has a solution in $L^2_{n,q}(\partial D, E)$ with norm C if and only if*

$$(4)\qquad \left|\int_D \langle f, \alpha\rangle\, dV\right|^2 \le C^2 \int_{\partial D} |\partial\rho \lrcorner \alpha|^2\, dS/|d\rho|^2$$

for all $(D'')^*$-closed $\alpha \in \mathscr{E}_{n,q+1}(\overline{D}, E)$.

Proof. First suppose there is such a solution u considered as a current such that $\bar\partial\rho\wedge u = 0$, cf., the remark above. Then $\int_D\langle f, \alpha\rangle\, dV = \int_{\partial D}\langle u, \partial\rho \lrcorner \alpha\rangle\, dS/|d\rho|$ so by Schwarz inequality,

$$\left|\int_D\langle f, \alpha\rangle\, dV\right|^2 \le \int_{\partial D}|u|^2\, dS/|d\rho| \int_{\partial D}|\partial\rho \lrcorner \alpha|^2\, dS/|d\rho|$$

which implies (4). For the converse, assume that (4) holds. Define the linear functional $\Lambda(\partial\rho \lrcorner \alpha) = \int_D\langle f, \alpha\rangle\, dV$ on E-valued (n, q)-forms on ∂D of the form $\partial\rho \lrcorner \alpha$ where $(D'')^*\alpha = 0$ in D. By (4) it is well defined and L^2-bounded, so there is a u such that $\int_{\partial D}|u|^2/|d\rho|\, dS \le C^2$ and (2) holds. □

In the next paragraph we shall derive an equality which gives a possibility to obtain estimates like (4).

3. A Bochner-Kodaira-Nakano-Morrey-Kohn-Hörmander type equality

In the notation from §1, the Bochner-Kodaira-Nokano identity is

(1) $$(D'')^*D'' + D''(D'')^* = (D')^*D' + D'(D')^* + i[\Theta, \Lambda]$$

where Λ is inner multiplication with the fundamental form ω, Θ is the curvature tensor on E, i.e. $\Theta = D^2$, and $[\ ,\ ]$ denotes commutator.

If X is compact (so that no boundary terms occur) (1) implies the estimate $\|D''\xi\|^2 + \|(D'')^*\xi\|^2 \ge (i[\Theta, \Lambda]\xi, \xi)$. If $a = i[\Theta, \Lambda]$ happens to be nonnegative on E-valued (p, q)-forms and f is a D''-closed (p, q)-form one gets [also using a local regularity result for the elliptic operator $\bar\Box = D''(D'')^* + (D'')^*D''$] the estimate

$$\left|\int\langle f, \xi\rangle\, dV\right|^2 \le \int\langle a^{-1}f, f\rangle\, dV \int|(D'')^*\xi|^2\, dV \quad \text{for all } \xi \in \mathscr{E}_{p,q}(X, E),$$

which means that there is a solution to $D''u = f$ with $\|u\|^2 \le \int(a^{-1}f, f)\, dV$, provided the right-hand side is finite.

In a domain D with boundary, one leads to study the D''-Neumann problem and here the starting point is the Morrey-Kohn-Hörmander identity. We will derive it below from (1). To deal with the $\bar\partial_b$-equation on $\bar D$ we need still another equality (Proposition 7) first found and used in [2] in the case of $(0, 1)$-forms (see the remark below) and trivial bundle. This one too will be derived from (1).

We first note how the various geometrical objects are affected if our original metric $\langle\ ,\ \rangle$ on E is modified.

Proposition 3. *If $\langle\ ,\ \rangle$ is changed to $\langle\ \rangle e^{-\varphi}$, then by obvious use of the index φ,*

(2) $$(D'')^{*\varphi} = (D'')^* + \partial\varphi \lrcorner,$$

(3) $$D'_\varphi = D' - \partial\varphi\wedge,$$

(4) $$\Theta_\varphi = \Theta + \partial\bar\partial\varphi,$$

and

(5) $$(D'_\varphi)^{*\varphi} = (D')^*.$$

Any of these follows from well-known identities, see e.g. [8], or by simple computations.

Now put $\varphi = t\log(-1/\rho)$ in $D = \{\rho < 0\}$ so that $\exp(-\varphi) = (-\rho)^t$ and $\partial\varphi = O(-1/\rho)$. Also put $(\ ,\ )_\varphi = \int_D\langle\ ,\ \rangle e^{-\varphi}\,dV$. If $t > 2$, we can, cf. (2), ..., (5), integrate by parts and obtain

$$\|D''\alpha\|_\varphi^2 + \|(D'')^{*\varphi}\alpha\|_\varphi^2 = \|D'_\varphi\alpha\|_\varphi^2 + \|(D'_\varphi)^{*\varphi}\alpha\|_\varphi^2 + i([\Theta_\varphi, \Lambda]\alpha, \alpha)_\varphi \tag{6}$$

from (1). Our next task is to compute the various terms in (6). We assume that $\alpha \in \mathscr{E}_{n,q}(\overline{D}, E)$ so that $D'_\varphi\alpha = 0$ and $[\Theta_\varphi, \Lambda]\alpha = \Theta_\varphi\Lambda\alpha$. Since, by (2),

$$(D'')^{*\varphi} = (D'')^* + \partial\varphi\,\lrcorner = (D'')^* - t(\partial\rho/\rho)\lrcorner$$

we get

$$\begin{aligned}\|(D'')^{*\varphi}\alpha\|_\varphi^2 &= \int_D(-\rho)^t|(D'')^*\alpha|^2\,dV \\ &\quad + 2t\,\mathrm{Re}\int_D(-\rho)^{t-1}\langle(D'')^*\alpha, \partial\rho\lrcorner\alpha\rangle\,dV + t^2\int_D(-\rho)^{t-2}|\partial\rho\lrcorner\alpha|^2\,dV.\end{aligned}\tag{7}$$

By (4),

$$\Theta_\varphi = \Theta - t\partial\overline{\partial}\rho/\rho + t\partial\rho\wedge\overline{\partial}\rho/\rho^2. \tag{8}$$

We need also

Lemma 4. *If* $\alpha \in \mathscr{E}_{n,q}(\overline{D}, E)$, *then*

$$i\langle\partial\rho\wedge\overline{\partial}\rho\wedge\Lambda\alpha, \alpha\rangle = |\partial\rho\lrcorner\alpha|^2. \tag{9}$$

Lemma 5. *If* $\psi \in \mathscr{E}(\overline{D})$, *then*

$$\int_D(-\rho)^{t-1}\psi\,dV \to \int_{\partial D}\psi\,dS/|d\rho|$$

when $t \searrow 0$.

Taking these for granted for the moment we get from (6), (7), (8) and (9) that

$$\begin{aligned}&\int_D(-\rho)^t|D''\alpha|^2\,dV + \int_D(-\rho)^t|(D'')^*\alpha|^2\,dV \\ &\quad + 2t\,\mathrm{Re}\int_D(-\rho)^{t-1}\langle(D'')^*\alpha, \partial\rho\lrcorner\alpha\rangle\,dV \\ &\quad + t^2\int_D(-\rho)^{t-2}|\partial\rho\lrcorner\alpha|^2\,dV \\ &= \int_D(-\rho)^t|(D')^*\alpha|^2\,dV + \int_D(-\rho)^t i\langle\Theta\Lambda\alpha, \alpha\rangle\,dV \\ &\quad + t\int_D(-\rho)^{t-1}\langle i\partial\overline{\partial}\rho\wedge\Lambda\alpha, \alpha\rangle\,dV + t\int_D(-\rho)^{t-2}|\partial\rho\lrcorner\alpha|^2\,dV\end{aligned}\tag{10}$$

for $t > 2$ and $\alpha \in \mathscr{E}_{n,q}(\overline{D}, E)$.

If we now assume that $D''\alpha = (D'')^*\alpha = 0$, combine the last terms on each side of equality (10) and let $t \searrow 1$ we get

Proposition 6. *Suppose* $\alpha \in \mathscr{E}_{n,q}(\overline{D}, E)$ *and* $D''\alpha = (D'')^*\alpha = 0$. *Then*

$$(11)\qquad \begin{aligned} i\int_D (-\rho)\langle \Theta\Lambda\alpha, \alpha\rangle\, dV + i\int_D \langle \partial\overline{\partial}\rho\Lambda\alpha, \alpha\rangle\, dV \\ + \int_D (-\rho)|(D')^*\alpha|^2\, dV = \int_{\partial D} |\partial\rho \,\lrcorner\, \alpha|^2\, dS/|d\rho|. \end{aligned}$$

Remark. Suppose $D \subset \mathbb{C}^n$. If $\langle\ ,\ \rangle$ is the metric $e^{-\psi}$ on the trivial line bundle (so that $\Theta = \partial\overline{\partial}\psi$) and $(n, 1)$-forms are identified with $(0, 1)$-forms in the obvious way, then (11) is exactly Proposition 5 in [2].

In a similar way we can also obtain the Morrey-Kohn-Hörmander identity.

Proposition 7. *If* $\alpha \in \mathscr{E}_{n,q}(\overline{D}, E)$ *and* $\partial\rho \,\lrcorner\, \alpha|_{\partial D} = 0$, *then*

$$(12)\qquad \begin{aligned} \int_D |D''\alpha|^2\, dV + \int_D |(D'')^*\alpha|^2\, dV = \int_D |(D')^*\alpha|^2\, dV \\ + i\int_{\partial D} \langle \partial\overline{\partial}\rho\Lambda\alpha, \alpha\rangle\, dS/|d\rho| + i\int_D \langle \Theta\Lambda\alpha, \alpha\rangle\, dV. \end{aligned}$$

Proof. By assumption $\partial\rho \,\lrcorner\, \alpha = O(-\rho)$ so (12) follows from (10) when $t \searrow 0$. □

We conclude this paragraph with proofs of the lemmas.

Proof of Lemma 4. Fix a point and $(1, 0)$-forms $\omega_1, \dots, \omega_n$ such that $\omega = \sum \omega_j \wedge \overline{\omega}_j$ and $\partial\rho = \omega_1$ at this point. We can write $\alpha = \alpha' + \alpha'' = \omega_1 \wedge \overline{\omega}_1 \wedge \gamma + \alpha''$, such that γ and α'' do not contain $\overline{\omega}_1$. Then $\omega_1 \wedge \overline{\omega}_1 \wedge \alpha = \alpha'$ and $\langle \alpha', \alpha\rangle = \langle \alpha', \alpha'\rangle$. On the other hand also, $|\omega_1 \,\lrcorner\, \alpha|^2 = |\omega_1 \,\lrcorner\, \omega_1 \wedge \overline{\omega}_1 \wedge \gamma|^2 = |\omega_1 \wedge \gamma|^2 = |\overline{\omega}_1 \wedge \omega_1 \wedge \gamma|^2 = |\alpha'|^2$ since $\overline{\omega}_1 \wedge \gamma$ does not contain ω_1. This proves the lemma. □

Proof of Lemma 5. We may assume that ψ has support in some small neighborhood of a boundary point and we let α be a $(2n-1)$-form such that $dV = d\rho \wedge \alpha/|d\rho|$ there. Then

$$\begin{aligned} t\int_D (-\rho)^{t-1}\psi\, dV &= t\int_D (-\rho)^{t-1} d\rho \wedge \psi\alpha/|d\rho| \\ &= -\int_D d(-\rho)^t \wedge \psi\alpha/|d\rho| = \int_D (-\rho)^t \wedge d(\psi\alpha/|d\rho|) \\ &\to \int_D d(\psi\alpha/|d\rho|) = \int_{\partial D} \psi\alpha/|d\rho| = \int_{\partial D} \psi\, dS/|d\rho| \end{aligned}$$

where we have used Stokes' theorem twice. □

4. A solution of the $\overline{\partial}_b$-equation

In this paragraph $D = \{\rho < 0\}$ is pseudoconvex and ρ is a C^2 plurisubharmonic defining function. Suppose that the hermitian operator (see [8])

$$A = i(-\rho)\Theta\Lambda + i\partial\overline{\partial}\rho\Lambda$$

is semipositive on E-valued $(n, q+1)$-forms, i.e.

$$\langle A\alpha, \alpha\rangle \geq 0, \qquad \alpha \in \mathscr{E}_{n,q+1}(\overline{D}, E).$$

Then Proposition 6 in §3 provides the estimate

$$\int_D \langle A\alpha, \alpha\rangle\, dV \le \int_{\partial D} |\partial\rho \lrcorner \alpha|^2\, dS/|d\rho|$$

for $\alpha \in \mathscr{E}_{p,q+1}(\overline{D}, E)$ such that $D''\alpha = (D'')^*\alpha = 0$.

If $f \in \mathscr{E}_{p,q+1}(\overline{D}, E)$ is $\overline{\partial}$-closed, we thus get

$$\left|\int_D \langle f, \alpha\rangle\, dV\right|^2 \le \int_D \langle A^{-1}f, f\rangle\, dV \int_{\partial D} |\partial\rho \lrcorner \alpha|^2\, dS/|d\rho| \tag{1}$$

for $\alpha \in \mathscr{E}_{p,q+1}(\overline{D}, E)$ such that $D''\alpha = (D'')^*\alpha = 0$. In order to ease the condition that $D''\alpha = 0$ in (1), we first, for simplicity, assume that ∂D is strictly pseudoconvex. Then, $|\alpha|^2 \le C\langle i\partial\overline{\partial}\rho\Lambda\alpha, \alpha\rangle$, for α such that $\partial\rho \lrcorner \alpha = 0$ on ∂D and since $|\langle\Theta\Lambda\alpha, \alpha\rangle| \le C|\alpha|^2$ (recall that E is assumed to be a bundle over X) we get from Proposition 7 the Basic Estimate

$$\int_D |(D')^*\alpha|^2\, dV + \int_{\partial D} |\alpha|^2\, dS \le C\left[\int_D |D''\alpha|^2\, dV + \int_D |(D'')^*\alpha|^2\, dV + \int_D |\alpha|^2\, dV\right]$$

if $\partial\rho \lrcorner \alpha|_{\partial D} = 0$. This ensures, see [6], regularity for the D''-Neumann problem and then any $\alpha \in \mathscr{E}_{n,q+1}(\overline{D}, E)$ has a smooth orthogonal decomposition $\alpha = \alpha' + \alpha''$ where $\overline{\partial}\alpha' = 0$ and α'' is orthogonal to $\overline{\partial}$-closed E-valued forms. In particular, $(D'')^*\alpha'' = 0$ and $\partial\rho \lrcorner \alpha''|_{\partial D} = 0$, cf. the remark after Proposition 1. Thus, if $(D'')^*\alpha = 0$ then $(D'')^*\alpha' = D''\alpha' = 0$ so (1) applies to α' and we hence obtain (1) for α as well. If ∂D is just pseudoconvex, we can still decompose $\alpha = \alpha' + \alpha''$ as before. Since then $\alpha'' \in \operatorname{Dom}(D'')^* \cap \operatorname{Dom} D''$ (in the densely defined operator sense) there are, by Proposition 2.1.1 in [9], $\alpha''_j \in \mathscr{E}_{n,q}(\overline{D}, E) \cap \operatorname{Dom}(D'')^*$ such that $\alpha''_j \to \alpha''$ in graph norm. In particular $\partial\rho \lrcorner \alpha''_j|_{\partial D} = 0$. If $\alpha'_j = \alpha - \alpha''_j$, then $\alpha'_j \to \alpha'$ in graph norm and $\partial\rho \lrcorner \alpha_{\partial D} = \partial\rho \lrcorner \alpha'_j|_{\partial D}$. Since also $(D'')^*\alpha''_j \to 0$, $D''\alpha''_j \to 0$ and $\int\langle f, \alpha'_j\rangle \to \int\langle f, \alpha\rangle$ one can proceed as before, but instead using the variant of (11) in which $(D'')^*\alpha$ and $D''\alpha$ are not supposed to vanish, cf. (10). By Proposition 2 we then have proved

Theorem 2. *Let $E \to X$ be a hermitian holomorphic vector bundle over the Kähler manifold X, and let $D = \{\rho < 0\}$ be a pseudoconvex relatively compact domain and ρ a C^2 plurisubharmonic defining function. Also suppose that $A = i(-\rho)\Theta\Lambda + i\partial\overline{\partial}\rho\Lambda$ is semipositive on E-valued $(n, q+1)$-forms. If $f \in \mathscr{E}_{n,q+1}(\overline{D}, E)$ is D''-closed, then there is a solution to $\overline{\partial}_b u = f$ in $L^2_{n,q}(\partial D, E)$ such that*

$$\int_{\partial D} |\partial\rho \lrcorner u|^2\, dS/|\partial\rho|^3 \le \int_D \langle A^{-1}f, f\rangle\, dV.$$

We recall that a bundle E is called Nakano semipositive if $\langle\Theta\Lambda\alpha, \alpha\rangle \ge 0$ for all $\alpha \in \mathscr{E}_{n,1}(X, E)$.

Corollary. *Suppose $E \to X$ is Nakano semipositive, $i\partial\overline{\partial}\rho \ge \delta I$ in D, and ψ is smooth and plurisubharmonic. Then if $f \in \mathscr{E}_{n,1}(\overline{D}, E)$ is $\overline{\partial}$-closed, there is* *a solution* u *to* $\bar{\partial}_b u = f$ *such that*

$$\int_{\partial D} |u|^2 e^{-\psi}\, dS/|d\rho| \le \frac{1}{\delta}\int_D |f|^2 e^{-\psi}\, dV. \tag{2}$$

In particular, one can let E be the trivial bundle over a domain D in $\mathbb{C}^n$ and thus get (2) for $(0, 1)$-forms f.

5. The division problem

When proving Theorem 1, we assume that q and g are holomorphic in a neighborhood of $\overline{D}$. The general case then follows by a normal family argument on the solutions in $D_\varepsilon = \{\rho < -\varepsilon\}$, since it turns out that the occurring constants only depend on derivatives up to second order of ρ near ∂D.

Remark. It is proved in [4] that any C^2 pseudoconvex domain admits a C^2-defining function ρ such that $-(-\rho)^\eta$ is (strictly) plurisubharmonic in D for some $\eta > 0$. Unfortunately, by our method the constants belonging to $D_\varepsilon = \{\rho < -\varepsilon\}$ seem to be unbounded when $\varepsilon \to 0$ so we cannot prove Theorem 1 in this general case. □

We thus have to consider the following situation. An exact sequence $0 \to S \xrightarrow{j} E \xrightarrow{g} Q \to 0$ of hermitian holomorphic vector bundles over X, such that j and g are holomorphic, S is equipped with the metric induced from E, $|g| \le 1$ and $\det gg^* \ge \delta^2 > 0$. The problem then is if for any holomorphic $q \in \mathscr{E}_{n,0}(\overline{D}, Q)$ there is a holomorphic solution $u \in \mathscr{E}_{n,0}(D, S)$ to

$$gu = q \tag{1}$$

such that

$$\int_{\partial D} |u|^2\, dS \le C^2 \int_{\partial D} |q|^2\, dS. \tag{2}$$

To find such holomorphic solutions, we proceed as follows. First we note that the pointwise minimal solution $\gamma q = g^*(gg^*)^{-1}q$ satisfies (2). Moreover, $\bar{\partial}(\gamma q) = (\bar{\partial}\gamma)q$ is a $\bar{\partial}$-closed $(n, 1)$-form with values in S, since $g\bar{\partial}\gamma q = \bar{\partial}(g\gamma q) = \bar{\partial} q = 0$. The hard step then is to find a $v \in L^2_{n,0}(\partial D, S)$ satisfying (2) such that $\bar{\partial}_b v = (\bar{\partial}\gamma)q$ in S, i.e. such that

$$\int_{\partial D} \langle v, \partial\rho \lrcorner \xi\rangle\, dS/|d\rho| = \int_D \langle (\bar{\partial}\gamma)q, \xi\rangle\, dV \tag{3}$$

for all $\xi \in \mathscr{E}_{n,1}(\overline{D}, S)$ such that $(D'')^*_S \xi = 0$.

We now claim that (3) implies that actually $\bar{\partial}_b v = (\bar{\partial}\gamma)q$ in E. Taking this for granted for the moment we conclude that $\bar{\partial}_b(\gamma q - v) = 0$ in (the trivial bundle) E which means that $u = \gamma q - v$ satisfies the tangential Cauchy-Riemann equation weakly.

This implies that u is the boundary values of a $U \in H^2(D)$ with norm

$$\|U\|^2_{H^2} = \int_{\partial D} |u|^2\, dS = \int_{\partial D} |v|^2\, dS + \int_{\partial D} |\gamma q|^2\, dS$$

(v and γq being orthogonal) and since $gU = q$ on ∂D, it must hold in D.

Thus our problem is solvable if (and only if) we can obtain (3) such that the $L^2(\partial D)$-norm of v is controlled by $(\int_{\partial D}|q|^2\,dS)^{1/2}$. By Proposition 2 this amounts to verify the inequality

$$(4)\qquad \left|\int_D \langle\overline{\partial}\gamma q\,,\,\xi\rangle\,dV\right|^2 \leq C_\delta^2\int_{\partial D}|q|^2\,dS\int_{\partial D}|\partial\rho\,\lrcorner\,\xi|^2\,dS/|\partial\rho|^2$$

for all $\xi\in\mathscr{E}_{n,1}(\overline{D}, S)$ such that $(D'')^*\xi=0$.

When trying to prove the estimate (4) one encounters two main difficulties. Firstly, although E and Q are trivial bundles with trivial metrics in our case, S acquires negative curvature which must be taken care of. However, it turns out that the curvature on S becomes nonnegative if the original metric is modified by a factor $e^{-\varphi}$, where φ is a *bounded* plurisubharmonic function. Since e^{φ} is bounded, it does not affect the estimates in any essential way. Secondly, even if we forget about the curvature problems, i.e. consider the scalar-valued case, an essential difficulty remains. As was mentioned in the introduction, one cannot use Theorem 2 directly since we must use more information about the right-hand side in our $\overline{\partial}_b$-equation than just a size estimate. Here the Wolff trick comes into play. Restricted to the scalar-valued case, our "Wolff theorem" can be stated

Proposition 8. *Suppose $f\in\mathscr{E}_{n,1}(\overline{D})$ is $\overline{\partial}$-closed, and that there is a bounded plurisubharmonic φ on $\overline{D}$ such that $|\langle f\,,\,\alpha\rangle|^2\leq\langle i\partial\overline{\partial}\varphi\Lambda\alpha\,,\,\alpha\rangle$ and*

$$\sum_{k=1}^n\left|\left\langle\frac{\partial f}{\partial z_k}\,,\,\alpha\right\rangle\right|^2\leq\Delta\varphi\langle i\partial\overline{\partial}\varphi\Lambda\alpha\,,\,\alpha\rangle$$

for all $(n\,,1)$-forms α. Then for any holomorphic q there is a solution $v\in L^2_{n,0}(\partial D)$ to $\overline{\partial}_b v=fq$ such that

$$\int_{\partial D}|v|^2\,dS\leq C^2\int_{\partial D}|q|^2\,dS,$$

where C only depends on $\|\varphi\|_\infty$ and D.

This proposition will be proved implicitly in the next paragraph. In the unit disc the assumption in the proposition is essentially, see [2], that $(1-|\zeta|^2)|f|^2$ and $(1-|\zeta|^2)|\partial f/\partial z|$ be Carleson measures, and the conclusion of the proposition implies, cf. the introduction, that there is a bounded solution.

Proof of the claim above. We actually have to verify that if $v\in L^2_{n,0}(\partial D, S)$ and (3) holds for all $\xi\in\mathscr{E}_{n,1}(\overline{D}, S)$ such that $(D'')^*_S\xi=0$, then it also holds for all $\alpha\in\mathscr{E}_{n,1}(\overline{D}, E)$ such that $(D'')^*_E\alpha=0$. However, if $p\colon E\to S$ is the orthogonal projection, then clearly (3) holds for $\xi=\alpha-p\alpha$. Moreover, if $(D'')^*_E\alpha=0$, then for any compactly supported $\eta\in\mathscr{E}_{n,1}(\overline{D}, S)$,

$$0=\int_D\langle(D'')^*_E\alpha\,,\,\eta\rangle=\int_D\langle\alpha\,,\,\overline{\partial}\eta\rangle=\int_D\langle p\alpha\,,\,\overline{\partial}\eta\rangle=\int_D\langle(D'')^*_S p\alpha\,,\,\eta\rangle,$$

so that $(D'')^*_S p\alpha=0$. Hence (3) holds for $\alpha=p\alpha+(\alpha-p\alpha)$. □

6. Proofs of Theorems 1 and 1′

Let β be the element in $\mathscr{E}_{1,0}(\overline{D}, \operatorname{Hom}(S, Q))$ such that its adjoint, with respect to the quotient metric on Q, $\beta^* \in \mathscr{E}_{0,1}(\overline{D}, \operatorname{Hom}(Q, S))$ equals $-\overline{\partial}\gamma$. Thus our equation to be solved ((3) in §5) becomes $\overline{\partial}_b v = -\beta^* q$. Moreover, since E has no curvature,

$$\Theta_S = \beta^* \wedge \beta \tag{1}$$

and since Q has no curvature,

$$-i\langle \beta^* \wedge \beta \Lambda \xi, \xi\rangle \le r\langle i\partial\overline{\partial}\psi\Lambda\xi, \xi\rangle, \qquad \xi \in \mathscr{E}_{n,1}(\overline{D}, S), \tag{2}$$

where $r = \min(n, \operatorname{rank} S)$ and $\psi = \log\det gg^*$ (note that g^* depends on the metric on Q). We also need the estimate

$$|\langle \beta^* q, \xi\rangle|^2 \le -|g^*(gg^*)^{-1}q|^2 i\langle \beta^* \wedge \beta\Lambda\xi, \xi\rangle \tag{3}$$

for $q \in \mathscr{E}_{n,0}(\overline{D}, Q)$ and $\xi \in \mathscr{E}_{n,1}(\overline{D}, S)$. For proofs of (1), (2) and (3) we refer to [12].

Recall (Proposition 6 in §3) that

$$\int_D (-\rho)\langle i(\Theta_S + \partial\overline{\partial}\varphi)\Lambda\alpha, \alpha\rangle e^{-\varphi}\, dV + \int_D \langle i\partial\overline{\partial}\rho\Lambda\alpha, \alpha\rangle e^{-\varphi}\, dV$$
$$+ \int_D (-\rho)|(D')^*\alpha|^2 e^{-\varphi}\, dV = \int_{\partial D} |\partial\rho \,\lrcorner\, \alpha|^2 e^{-\varphi}\, dS/|d\rho|$$

for $\alpha \in \mathscr{E}_{n,1}(\overline{D}, S)$ if $\overline{\partial}\alpha = (D'')^{*\varphi}\alpha = 0$.

Hence if $\varphi = (r+\varepsilon)\psi$ and ρ is plurisubharmonic, we get by (1) and (2)

$$\int_D (-\rho)|(D')^*\alpha|^2 e^{-\varphi}\, dV \le \int_{\partial D} |\partial\rho \,\lrcorner\, \alpha|^2 e^{-\varphi}\, dS/|d\rho| \tag{4}$$

and

$$\int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda\alpha, \alpha\rangle e^{-\varphi}\, dV \le \frac{r+\varepsilon}{r}\int_{\partial D} |\partial\rho \,\lrcorner\, \alpha|^2 e^{-\varphi}\, dS/|d\rho|. \tag{5}$$

From (3) we also get

$$|\langle \beta^* q, \xi\rangle|^2 \le |g^*(gg^*)^{-1}q|^2 \frac{r}{r+\varepsilon}\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle, \tag{6}$$

for $\xi \in \mathscr{E}_{n,1}(\overline{D}, S)$ and $q \in \mathscr{E}_{n,0}(\overline{D}, Q)$. We now claim that Theorem 1 follows from

Proposition 9. *If D and ρ are as in Theorem 1, then*

$$\left|\int_D \langle \beta^* q, \xi\rangle\, dV\right|^2 \le C_\delta^2 \int_{\partial D} |q|^2\, dS \int_{\partial D} |\partial\rho \,\lrcorner\, \xi|^2 e^{\varphi}\, dS \tag{7}$$

for any holomorphic $q \in \mathscr{E}_{n,0}(\overline{D}, Q)$ and $\xi \in \mathscr{E}_{n,1}(\overline{D}, S)$ such that $(D'')^{*\varphi}(e^\varphi\xi) = \overline{\partial}(e^\varphi\xi) = 0$.

Here C_δ is the constant described in Theorem 1.

Proof of Theorem 1. By the discussion in §5 it is enough to verify (4) in §5 for $\xi \in \mathscr{E}_{n,1}(\overline{D}, S)$ such that $(D'')^*\xi = 0$. Note that

$$(D'')^*\xi = 0 \quad \text{iff} \quad (D'')^{*\varphi}(e^\varphi\xi) = 0.$$

Putting $\alpha = e^\varphi\xi$, (7) then says that

$$(8)\qquad \left|\int_D \langle \beta^* q, \alpha\rangle e^{-\varphi}\, dV\right|^2 \le C_\delta^2 \int_{\partial D} |q|^2\, dS \int_{\partial D} |\partial\rho \rfloor \alpha|^2 e^{-\varphi}\, dS$$

for all α such that $\overline{\partial}\alpha = (D'')^{*\varphi}\alpha = 0$. As in §4, we can obtain (8) for all α with $(D'')^{*\varphi}\alpha = 0$. But this means that (7) holds for all ξ with $(D'')^*\xi = 0$. Finally $e^\varphi = (\det gg^*)^{r+\varepsilon} \le 1$ by assumption, and hence we have verified (4) in §5. □

Proof of Proposition 9. We consider g as a $j\times k$-matrix of holomorphic functions on $\overline{D}$ and use the norms

$$\|g\|^2 = \sum_{\tau\nu} |g_{\tau\nu}|^2, \quad |g'|^2 = \sum_{i\tau\nu} |\partial g_{\tau\nu}/\partial\zeta_i|^2.$$

The assumptions on g imply that

$$(9)\qquad |(gg^*)^{-1}| \lesssim 1/\delta^2,\ |g^*(gg^*)^{-1}q|^2 \lesssim (1/\delta^2)|q|^2.$$

If Δ is the $\mathbb{R}^{2n}$-Laplacian, then

$$(10)\qquad |g'|^2/|g|^{2-2\varepsilon} \le C_\varepsilon \Delta |g|^{2\varepsilon}, \qquad \varepsilon > 0.$$

We also need the inequalities

$$(11)\qquad \int_D (-\rho)|f'|^2\, dV \lesssim \int_{\partial D} |f|^2\, dS$$

and

$$(12)\qquad \int_D (-\rho)|f|^2 \Delta\psi e^\psi\, dV \lesssim \int_{\partial D} |f|^2 e^\psi\, dS$$

for holomorphic f and subharmonic ψ. They follow from Green's formula. Recall that $-\beta^* = \overline{\partial}\gamma = \overline{\partial}[g(gg^*)^{-1}]$ so that, for an S-valued ξ,

$$-\langle \beta^* q, \xi\rangle = \langle (\partial g)^*(gg^*)^{-1} q, \xi\rangle.$$

We have to estimate

$$I = \int_D \langle \beta^* q, \xi\rangle\, d\lambda,\ (D'')^{*\varphi}(e^\varphi\xi) = \overline{\partial}(e^\varphi \xi) = 0.$$

Let χ be a smooth function which is identically 1 near ∂D and such that $\partial\rho$ is nonvanishing on its support. If $L = \chi|\partial\rho|^{-2}\sum_1^n (\partial\rho/\partial\overline{\zeta}_j)\partial/\partial\zeta_j$, then we can write

$$I = \int_D (1-\chi)\langle \beta^* q, \xi\rangle\, dV - \int_D L(-\rho)\langle \beta^* q, \xi\rangle\, dV,$$

and an integration by parts in the second integral gives us

$$\begin{aligned} I = &\int_D (1-\chi)\langle \beta^* q, \xi\rangle\, dV + \int_D (-\rho) O(1)\langle \beta^* q, \xi\rangle\, dV \\ &+ \int_D (-\rho)\langle (L\beta^*) q, \xi\rangle dV + \int_D (-\rho)\langle \beta^* Lq, \xi\rangle\, dV \\ &+ \int_D (-\rho)\langle \beta^* q, (\overline{L}\varphi)\xi + \overline{L}\xi\rangle\, dV - \int_D (-\rho)\langle \beta^* q, \overline{L}\varphi\xi\rangle\, dV \\ = &\ \mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5 \end{aligned}$$

where $O(1)$ only depends on derivatives up to second order of ρ and χ, and where $\varphi = (r+\varepsilon)\log\det gg^*$. The proof is concluded by estimating each I_i. By (6) and Schwarz inequality we have that, for $i = 0, 1$,

$$|\mathrm{I}_i|^2 \lesssim \int_D (-\rho)|g^*(gg^*)^{-1}q|^2 e^{-\varphi}\, dV \int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi}\, dV.$$

Now, cf. (9),

$$\begin{aligned}\int_D (-\rho)|g^*(gg^*)^{-1}q|^2 e^{-\varphi}\, dV &\lesssim \frac{1}{\delta^2\delta^{2(r+\varepsilon)}}\int_D (-\rho)|q|^2\, dV \\ &\lesssim \left(\frac{1}{\delta^{1+r+\varepsilon}}\right)^2 \int_{\partial D}|q|^2\, dS\end{aligned}$$

since q is holomorphic (cf. (12) with e.g. $\psi = |\zeta|^2$). Also

$$\int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi}\, dV = \int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda(e^{\varphi}\xi), e^{\varphi}\xi\rangle e^{-\varphi}\, dV$$

and hence by (5), $\lesssim \int_{\partial D}|\partial\rho \lrcorner \xi|^2 e^{\varphi}\, dS$. Thus we have obtained the required estimate for I_0 and I_1. To handle I_2, first note that

$$\begin{aligned}-\langle(L\beta^*)q, \xi\rangle &= \langle(\partial g)^*(gg^*)^{-1}(Lg)g^*(gg^*)^{-1}q, \xi\rangle \\ &= \langle\beta^*(Lg)g^*(gg^*)^{-1}q, \xi\rangle\end{aligned}$$

so by (6) and (9),

$$\begin{aligned}|\langle(L\beta^*)q, \xi\rangle|^2 &\le |g^*(gg^*)^{-1}(Lg)g^*(gg^*)^{-1}q|^2 e^{-\varphi}\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi} \\ &\lesssim \frac{1}{\delta^2\delta^{2(r+\varepsilon)}\delta^{2\varepsilon}}\frac{|g'|^2}{(\det gg^*)^{1-\varepsilon}}|q|^2\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi}.\end{aligned}$$

If g is a row matrix, i.e. $j = 1$, then $\det gg^* = |g|^2$ so we can use (10) and get

$$\begin{aligned}|\mathrm{I}_2|^2 &\le \left(\frac{1}{\delta^{1+r+2\varepsilon}}\right)^2 \int_D (-\rho)|q|^2\Delta|g|^{2\varepsilon}e^{|g|^{2\varepsilon}} \int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi} \\ &\lesssim \left(\frac{1}{\delta^{1+r+2\varepsilon}}\right)^2 \int_{\partial D}|q|^2\, dS \int_{\partial D}|\partial\rho \lrcorner \xi|^2 e^{\varphi}\, dS.\end{aligned}$$

If $j > 1$ we estimate $1/\det gg^*$ by $1/\delta^2$ and use the simpler inequality $|g'|^2 \le \Delta|g|^2$. For simplicity we assume for the rest of the proof that $j = 1$. To handle I_3, we note that

$$|\langle\beta^* Lq, \xi\rangle|^2 \lesssim \left(\frac{1}{\delta^{1+r+\varepsilon}}\right)^2 |q'|^2\langle i\partial\overline{\partial}\varphi\Lambda\xi, \xi\rangle e^{\varphi}$$

and here the first factor is treated by (11) and the second one as before. Further, we have

$$\begin{aligned}|\mathrm{I}_4|^2 &\lesssim \int_D (-\rho)|\beta^* q|^2 e^{-\varphi}\, dV \int_D (-\rho)|(\overline{L}\varphi)\xi + \overline{L}\xi|^2 e^{\varphi}\, dV \\ &\lesssim \left(\frac{1}{\delta^{1+r+2\varepsilon}}\right)^2 \int_D (-\rho)\frac{|g'|^2}{|g|^{2-2\varepsilon}}|q|^2\, dV \int_D (-\rho)|\overline{L}(e^{\varphi}\xi)|^2 e^{-\varphi}\, dV.\end{aligned}$$

The first factor is handled as before and the second one is estimated by (4). Finally,

$$|\mathrm{I}_5|^2 \lesssim \left(\frac{1}{\delta^{1+r+\varepsilon}}\right)^2 \int_D (-\rho)|\overline{L}\varphi|^2|q|^2\,dV \int_D (-\rho)\langle i\partial\overline{\partial}\varphi\Lambda\xi\,,\,\xi\rangle e^{\varphi}\,dV.$$

Note that $|\overline{L}\varphi|^2 \leq |g'|^2/|g|^2$ so that the first factor is dominated by

$$\frac{1}{\delta^{2\varepsilon}}\int_D (-\rho)(\Delta|g|^{2\varepsilon})|q|^2 e^{|g|^{2\varepsilon}}\,dV \lesssim \frac{1}{\delta^{2\varepsilon}}\int_{\partial D}|q|^2\,dS$$

and hence the proposition is proved. □

Proof of Theorem 1′. It is enough to show that

$$\left|\int_D \langle\beta^* q\,,\,\xi\rangle\,dV\right|^2 \leq C_\varepsilon \int_{\partial D} e^{-\chi}\,dS \int_{\partial D} |\partial\rho \rfloor \xi|^2 e^{\varphi+\chi}$$

for all ξ such that $(D'')^{*\psi}(e^{\psi}\xi) = \overline{\partial}(e^{\psi}\xi) = 0$, where we have put $\psi = \varphi + \chi = (r+\varepsilon')\log|g|^2 + \chi$ and ε' is less than the ε in the hypothesis of Theorem 1′.

Then most arguments when estimating the left-hand side work as before. We have just two new difficulties. For the term I_3 we have

$$|\mathrm{I}_3|^2 \lesssim \int_D (-\rho)|Lq|^2/|g|^{2(1+r+\varepsilon')}e^{-\chi}\int_D (-\rho)\langle i\partial\overline{\partial}\psi\Lambda\xi\,,\,\xi\rangle e^{\psi}.$$

Since $|g|^{2(1+r+\varepsilon)} \geq |q|^2$, $|g|^{2(1+r+\varepsilon')} \geq |q|^{2-\varepsilon''}$ and hence the first factor is

$$\begin{aligned}
&\lesssim \int_D (-\rho)\Delta|q|^{\varepsilon''}e^{-\chi} \leq \int_D (-\rho)\Delta(|q|^{2''} - \chi)e^{|q|^{\varepsilon''}-\chi}\\
&\lesssim \int_{\partial D} e^{|q|^{\varepsilon''}-\chi}\,dS \lesssim \int_{\partial D} e^{-\chi}\,dS.
\end{aligned}$$

When estimating I_5 we show up with a factor

$$\begin{aligned}
&\int_D (-\rho)|q|^2|\nabla\psi|^2 e^{-\psi} \leq \int_D (-\rho)|g|^{\varepsilon''}|\nabla\psi|^2 e^{-\chi}\\
&\qquad \lesssim \int_D (-\rho)|g|^{\varepsilon''}|\nabla(\log|g|^2)|^2 e^{-\chi} + \int_D (-\rho)|\nabla\chi|^2 e^{-\chi} = a_1 + a_2.
\end{aligned}$$

But

$$a_1 \lesssim \int_D (-\rho)\Delta(|g|^{\varepsilon''} - \chi)e^{|g|^{\varepsilon''}-\chi} \lesssim \int_{\partial D} e^{-\chi}$$

and so is a_2. This concludes the proof of Theorem 1′. □

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